Ch-1 Vector Spaces

We have done the def^n of vector spaces and seen some examples. Recalling the def^n:

Def^n:- A <u>Vector space</u> V over a field F consists of a set on which two operations (addition and scalar multiplication) are defined so that for each pair x, y in V, there is a unique element $x+y$ in V and for each $\alpha \in F$ and each $x \in V$, αx is an unique element in V, such that the following conditions hold:

(VS1) for all x, y in V, $x+y = y+x$

(VS2) For all x, y & z in V,
$$(x+y)+z = x+(y+z)$$

(VS3) $\exists$ $0 \in V$ such that $x+0 = x = 0+x$ $\forall$ $x \in V$.

(VS4) for each element $x \in V$, $\exists$ $y \in V$ such that $x+y = y+x = 0$.

(VS5) $1.x = x$ $\forall$ $x \in V$.

(VS6) $\alpha(\beta x) = (\alpha\beta)x$ for all $\alpha, \beta \in F$ & $x \in V$

(VS 7) $\alpha(x+y) = \alpha x + \alpha y \quad \forall\, \alpha \in F \;\&\; x, y \in V$

(VS 8) $(\alpha+\beta)x = \alpha x + \beta x \quad \forall\, \alpha, \beta \in F \;\&\; x \in V.$

Examples:-

(1) Consider $\mathbb{R}^n$, then $\mathbb{R}^n$ is a Vector space over $\mathbb{R}$ under the operations: component wise addition and component wise scalar multiplication.

(2) $P_n(\mathbb{R})$ - set of all real polynomials upto degree n,

then $P_n(\mathbb{R})$ is a V.S. over $\mathbb{R}$ under operations defined as

let $f(x), g(x) \in P_n(\mathbb{R})$

$f(x) = a_0 + a_1 x + \dots + a_m x^m$

$g(x) = b_0 + b_1 x + \dots + b_l x^l$ let $m < l$.

$f(x) + g(x) = (a_0+b_0) + (a_1+b_1)x + \dots + (a_m+b_m)x^m + b_{m+1}x^{m+1} + \dots + b_l x^l$

and $\alpha f(x) = \alpha a_0 + \alpha a_1 x + \dots + \alpha a_m x^m$.

Thm 1.1 (Cancellation law):- Let V be a vector space over field F and if x, y and z are vectors in V such that $x+z = y+z$, then $x = y$.

Proof:- As $z \in V$, there exist a vector $v \in V$ such that $z + v = 0$. (Additive inverse of z).

$$\text{Then } x = x+0 = x+(z+v)$$
$$= (x+z)+v$$
$$= (y+z)+v$$
$$= y+(z+v) = y+0 = y.$$

$$\therefore \quad x = y. \quad \blacksquare$$

Remarks:-
As a vector space V is abelian group ~~as~~ under the addition, it can be easily shown that the 0 vector is unique and additive inverse of any vector in V is also unique. (Do it).

Thm 1.2:- Let V be a vector space over field F, then the following statements are true:

(a) $0x = 0$ for each $x \in V$.

(b) $(-a)x = -(ax) = a(-x)$ for each $a \in F, x \in V$.

(c) $a0 = 0$ for each $a \in F$.

Proof:- (a) Consider

$0x + 0x = (0+0)x$

$= 0x = 0x + 0$ (As $0+0=0$ in F).

$\Rightarrow 0x + 0x = 0x + 0$

$\therefore 0x = 0$ (By cancellation law)

(b) Consider,

$ax + (-a)x = [a+(-a)]x$

$= 0x = 0$

$\therefore ax + (-a)x = 0$ — ①

Also, $ax + (-ax) = 0$ ② [$-ax$ being additive inverse of ax in V]

Then from ① and ②, we get

$(-a)x = -(ax)$. — ③

In particular $(-1)x = -x$.

So, $a(-x) = a[(-1)x]$

$= [a(-1)]x = (-a)x$.

$\therefore a(-x) = (-a)x$. — ④

from ③ & ④,

$(-a)x = -(ax) = a(-x)$.

c) $a0 + a0 = a(0+0) = a0 = a0 + 0$

$\therefore a0 = 0$

Section 1.3.

Def^n:- (Subspaces):- A subset W of a vector ~~field~~ space V over a field F is called subspace of V if W is a vector space over F with the operations of addition and scalar multiplication of V.

Thm 1.3 (Subspace test)
Let V be a vector space and $W \subseteq V$. Then W is a subspace of V if and only if the following conditions hold for operations defined on V.

(i) $0 \in W$
(ii) $x+y \in W$ whenever $x, y \in W$.
(iii) $cx \in W$ whenever $c \in F$, $x \in W$.

Proof:- Suppose W is subspace of V, then W is vector space over F, so W satisfies all three conditions (i), (ii) & (iii). (check).

Conversely, suppose W satisfies (i), (ii) & (iii).

T.S.:- W is subspace of V.

Condition (ii) & (iii) implies that the operation of V are defined (binary) on W.

Now we just need to show W satisfies 8 conditions of vector space.

Let $x, y \in W$
then $x, y \in V$
So $x+y = y+x$ (As V is vector space)
$\therefore$ $x+y = y+x$ $\forall x, y \in W$.

Similarly, (VS 2), (VS 3), (VS 5), (VS 6), (VS 7), (VS 8) can be shown, we only need to show that additive inverse exists in W.

Let $x \in W$, then $(-1)x = -x \in W$ (By (iii))]
also $x + (-x) = 0$.
$\Rightarrow$ Additive inverse exists in W.
$\Rightarrow$ W satisfies (VS 4).
$\Rightarrow$ W is vector space over F with operations of V.
$\Rightarrow$ W is subspace of V.

(Note:- Here VS1, VS2, .., VS 8 conditions are defined in book on page no. 7).

Examples:-

(1.) Let S be the set of all diagonal 2x2 matrices.
then S is a subspace of $M_{2\times2}(R)$.
As $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$

Let $A, B \in S$

then $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ & $B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$

$$A+B = \begin{bmatrix} a_1+b_1 & 0 \\ 0 & a_2+b_2 \end{bmatrix} \in S.$$

and $$cA = \begin{bmatrix} ca_1 & 0 \\ 0 & ca_2 \end{bmatrix} \in S.$$

So, $A+B \in S$, $\forall A, B \in S$
and $cA \in S$, $\forall c \in \mathbb{R}$, $A \in S$.

$\Rightarrow$ S is a subspace of $M_{2\times 2}(\mathbb{R})$.

② Let W be the set of 3×3 symmetric matrices (i.e. $A^T = A$).
then W is subspace of $M_{3\times 3}(\mathbb{R})$.

As 0 matrix is symmetric matrix.
$\therefore$ $0 \in W$.

Also, sum of two symmetric matrix is symmetric $(\because (A+B)^T = A^T + B^T)$.

Using above please show W is subspace of $M_{3\times 3}(\mathbb{R})$.

Thm 1.4 Any intersection of subspaces of a vector space V is a subspace of V.

Proof:- Let W_i be the subspaces of V for $i \in I$, where I is an index set.

Let $W = \bigcap_{i \in I} W_i$

T.S. W is subspace of V.

As $0 \in W_i \quad \forall\, i \in I$.

$0 \in \bigcap_{i \in I} W_i = W$

$\therefore \quad 0 \in W$.

Let $x, y \in W$ then $x \in W_i$, $y \in W_i$ $\quad \forall\, i \in I$.

$\therefore \quad x+y \in W_i \quad \forall\, i \in I$.

$\Rightarrow \quad x+y \in \bigcap W_i = W$

$\therefore \quad x+y \in W$.

Similarly, $cx \in W$, $\forall\, c \in F$, $x \in W$.

$\therefore$ W is a subspace of V.

Q^n:- Check whether union of two subspace of V is again a subspace or not.

Section 1.4

Def^n:- Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is said to be linear combination of vectors of S if there exists a finite number of vectors $v_1, v_2, \dots, v_n$ in S and scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ in F such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$.

We say that v is linear combination of $v_1, v_2, \dots, v_n$ and call $\alpha_1, \alpha_2, \dots, \alpha_n$ the coefficients.

Examples:-

1. Let $\mathbb{R}^2$ be a vector space over $\mathbb{R}$, then any vector of $\mathbb{R}^2$ can be written as a linear combination of vectors $(1,0)$ and $(0,1)$.

   A. Let $(x,y) \in \mathbb{R}^2$
   then $(x,y) = x(1,0) + y(0,1)$.

2. Similarly, every vector of $\mathbb{R}^n$ is a linear combination of vectors of set S, where
   $S = \{(1,0,\dots,0), (0,1,\dots,0), \dots\dots (0,0,\dots,1)\}$

3. $2x^3 - 2x^2 + 12x - 6$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$ in $P_3(\mathbb{R})$.

$\rightarrow$ Let $\alpha, \beta \in \mathbb{R}$ and

$$2x^3 - 2x^2 + 12x - 6 = \alpha(x^3 - 2x^2 - 5x - 3) + \beta(3x^3 - 5x^2 - 4x - 9)$$

$$\Rightarrow 2x^3 - 2x^2 + 12x - 6 = (\alpha + 3\beta)x^3 + (-2\alpha - 5\beta)x^2 + (-5\alpha - 4\beta)x + (-3\alpha - 9\beta)$$

Comparing both sides, we get

$$\alpha + 3\beta = 2$$
$$-2\alpha - 5\beta = -2$$
$$-5\alpha - 4\beta = 12$$
$$-3\alpha - 9\beta = -6$$

Solving above system, we get

$$\alpha = -4 \ \& \ \beta = 2$$

So, $2x^3 - 2x^2 - 12x - 6 = -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9)$.

(4.) Check whether $(1, 2, -3)$ can be written as a linear combination of vectors $(-3, 2, 1)$ and $(2, -1, -1)$ in $\mathbb{R}^3$. – (Do it).

Defn:- Let S be a non-empty subset of a vector space V over a field F. The span of S, denoted by $span(S)$, is the set ~~of~~ consisting of all linear combinations of the vectors in S.

$$span(S) = \{\alpha_1 v_1 + \dots + \alpha_n v_n \mid \alpha_i \in F, \ v_i \in S, \ n \in \mathbb{N}\}.$$

and span of empty set is $\{0\}$. i.e. $span(\phi) = \{0\}$.

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Examples:

1. Let $S = \{(1,0),(0,1)\}$ in $\mathbb{R}^2$
then $span(S) = \mathbb{R}^2$. (verify).

2. Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
be the subset of $M_{2\times 2}(\mathbb{R})$.
then $Span(S) = M_{2\times 2}(\mathbb{R})$. (verify).

3. Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$
be the subset of $M_{2\times 2}(\mathbb{R})$.
then $span(S)$ = set of all symmetric 2×2 matrices.

$\rightarrow$ Let $sym(S)$ be the set of all symmetric 2×2 matrices

T.S.:- $span(S) = sym(S)$.

Let $A \in span(S)$

then $\exists\ \alpha_1, \alpha_2, \alpha_3$ in $\mathbb{R}$ such that

$$A = \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{bmatrix}$$

$\Rightarrow A \in sym(S)$

$\Rightarrow span(S) \subseteq sym(S)$ — ①

Now let $A \in Sym(S)$

then $A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix}$ (?)

where $a_1, a_2, a_4 \in \mathbb{R}$

then $A = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow$ $A \in Span(S)$

$\Rightarrow$ $Sym(S) \subseteq Span(S)$ - (2)

from eq. (1) and eq. (2),

$Span(S) = Sym(S)$.

Thm 1.5:- The span of any subset S of a vector space V over field F is a subspace of V. Moreover, any subspace V that contains S must also contain the span of S. (i.e. Span of S is the smallest subspace of V containing S)

Proof:- Let S be a non-empty subset of V

(What will happen if we take $S = \phi$).

T.S.:- $span(S)$ is subspace of V.

$\rightarrow$ Let $x, y \in span(S)$

then $x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

and $y = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_m w_m$

where $v_i, w_i \in V$ $\forall i$ & $\alpha_i, \beta_i \in F$.

then

$$x+y = \alpha_1 v_1 + \dots + \alpha_n v_n + \beta_1 w_1 + \dots + \beta_m w_n$$

$\Rightarrow$ $x+y$ is linear combination of $\{v_1, \dots, v_n, w_1, \dots, w_m\}$

$\Rightarrow$ $x+y \in \text{span}(S)$

Also let $\alpha \in F$

then $\alpha x = \alpha\alpha_1 v_1 + \dots + \alpha\alpha_n v_n$

$\Rightarrow$ $\alpha x \in \text{span}(S)$

Thus by subspace test, $\text{span}(S)$ is subspace of V.

T.S. :- $\text{span}(S)$ is smallest subspace of V containing S.

$\rightarrow$ Let W be any subspace containing S.

We want to show that $\text{span}(S) \subseteq W$.

Let $x \in \text{span}(S)$ be arbitrary.

then $x = \alpha_1 v_1 + \dots + \alpha_n v_n$

where $v_1, v_2, \dots, v_n \in S$ & $\alpha_1, \alpha_2, \dots, \alpha_n \in F$.

then $v_1, v_2, \dots, v_n \in W$

also, $\alpha_1 v_1, \alpha_2 v_2, \dots, \alpha_n v_n \in W$ (?)

$\Rightarrow$ $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in W$ (?)

$\Rightarrow$ $x \in W$

$\Rightarrow$ $\text{span}(S) \subseteq W$.

Def^n:- A subset S of a vector space V generates (or spans) V if $\text{span}(S) = V$.

Examples:-

① The vectors $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$ generates $\mathbb{R}^3$.

② Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ then S generates $\mathbb{R}^3$.

→ T.S.:- $\text{span}(S) = \mathbb{R}^3$

firstly, $\text{span}(S) \subseteq \mathbb{R}^3$ - (?) why.

We want to show that $\mathbb{R}^3 \subset \text{span}(S)$

Let $v \in \mathbb{R}^3$ then $v = (x_1, x_2, x_3)$

and $(x_1, x_2, x_3) = \alpha_1 (1,1,0) + \alpha_2 (1,0,1) + \alpha_3 (0,1,1)$

$\Rightarrow (x_1, x_2, x_3) = (\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3)$

Comparing both sides, we get

$$\alpha_1 + \alpha_2 = x_1$$
$$\alpha_1 + \alpha_3 = x_2$$
$$\alpha_2 + \alpha_3 = x_3$$

Solving above equations, we get

$$\alpha_1 = \frac{x_1 + x_2 - x_3}{2}, \quad \alpha_2 = \frac{x_1 - x_2 + x_3}{2}, \quad \alpha_3 = \frac{-x_1 + x_2 + x_3}{2}.$$

Hence, $v = (x_1, x_2, x_3) \in \text{span}(S)$ $\therefore$ S generates $\mathbb{R}^3$.

(3) Let $S = \cancel{\{1, x, x^2\}}$ $S = \{1, x, x^2, x^3\}$, then $P_3(\mathbb{R})$ (- set of real valued polynomials upto degree 3) is generated by S.

firstly, check that $\text{span}(S) \subseteq P_3(\mathbb{R})$

Now let $f(x) \in P_3(\mathbb{R})$

then $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

and $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

then $f(x) = a_0(1) + a_1(x) + a_2(x^2) + a_3(x^3)$

$\Rightarrow$ $f(x) \in \text{Span}(S)$

$\therefore$ $P_3(\mathbb{R}) \subseteq \text{span}(S)$

Hence $\text{Span}(S) = P_3(\mathbb{R})$.

Q15 Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_1) \cap \text{Span}(S_2)$.

$\rightarrow$ Let $x \in \text{Span}(S_1 \cap S_2)$

then there exists $v_1, v_2, \ldots, v_n \in S_1 \cap S_2$

and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that

$x = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

And as $v_i \in S_1 \cap S_2$ for $i = 1, 2, \ldots, n$

$\Rightarrow$ $v_i \in S_1$ and $v_i \in S_2$

$\Rightarrow$ $x \in \text{Span}(S_1)$ and $x \in \text{Span}(S_2)$

$\therefore$ $x \in \text{Span}(S_1) \cap \text{Span}(S_2)$

Thus, $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_1) \cap \text{Span}(S_2)$.

Q Check whether $Span(S_1) \cap Span(S_2) \subseteq Span(S_1 \cap S_2)$ in Q15. If yes prove, otherwise give a counter example.

Section- 1.5

As a vector space V can be generated by many subsets of V, to find the smallest subset of V which generates V, we use the concept of linearly dependence & linearly independence.

Def^n:- A subset S of a vector space V is called linearly dependent if there exist a finite no. of distinct vectors $v_1, v_2, \ldots, v_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

In this case, we say that vectors of S are linearly dependent.

Example:- ① Consider

$$S = \{(1,0,1), (2,0,2), (1,1,0)\} \text{ in } \mathbb{R}^3$$

then S is linearly dependent.

As let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ s.t

$$\alpha_1(1,0,1) + \alpha_2(2,0,2) + \alpha_3(1,1,0) = 0$$

then, $\alpha_1 + 2\alpha_2 + \alpha_3 = 0$

$$\alpha_3 = 0$$

$$\alpha_1 + 2\alpha_2 = 0$$

Solving above we get, $\alpha_3 = 0$ & $\alpha_1 + 2\alpha_2 = 0$

So one of the solution is $\alpha_1 = 2$, $\alpha_2 = -1$, $\alpha_3 = 0$

Thus, S is linearly dependent.

(2) Let $S = \{(1,0), (0,1)\}$ in $\mathbb{R}^2$.

then S is not linearly dependent.

As let $\alpha_1, \alpha_2 \in \mathbb{R}$ s.t.

$$\alpha_1(1,0) + \alpha_2(0,1) = 0$$

$$(\alpha_1, \alpha_2) = 0$$

$$\Rightarrow \alpha_1 = 0 \quad , \alpha_2 = 0$$

But one of the α_1 or α_2 has to be nonzero for S to be a linearly dependent subset.

<u>Defn</u>:- A subset S of a vector space that is not linearly dependent is called <u>linearly independent</u>. By that we mean, if there exists a finite no. of distinct vectors $v_1, v_2, \ldots, v_n$ in S, then for

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

all scalars a_i has to be zero for $i = 1, 2, \ldots, n$.

Examples:-

(1) Consider $S = \{(1,0), (0,1)\}$ in $\mathbb{R}^2$.
then S is linearly independent.

(2) Consider $S = \{1, x, x^2, x^3\}$ in $P_3(R)$.
then S is linearly independent.

(3) Consider $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ in R^3.
Check whether S is linearly dependent
or linearly independent. (Do it).

Remarks:-

(1) The empty set is linearly dependent.

(2) Set consisting of a single zero element i.e.
$S = \{0\}$, then S is linearly dependent.

(3) set consisting of a single nonzero element
is linearly independent.

(4) A set is linearly independent if and only
if the only representations of 0 as linear
combination of its vector are trivial representation.

Thm 1.6:- Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Proof:- As S_1 is linearly dependent

$\Rightarrow$ there exists finite vectors $v_1, v_2, \ldots, v_n$ in S_1 and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

and as $v_1, v_2, \ldots, v_n \in S_1$

$\Rightarrow$ $v_1, v_2, \ldots, v_n \in S_2$ let $v_{n+1}, \ldots, v_m \in S_2$ for that take $\alpha_i = 0$.

and $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$

$\Rightarrow$ S_2 is linearly dependent.

Corollary:- Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Proof:- Suppose on contrary, S_1 is linearly dependent. then by above theorem S_2 is also linearly dependent, which gives contradiction.

So, S_1 is linearly independent.

Q → Try proving above corollary without using above theorem 1.6.

Thm 1.7:- Let S be a linearly independent subset of a vector space V and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$.

Proof:- Let $v \in span(S)$

then there exists $v_1, v_2, \ldots, v_n$ in S and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

$$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + (-1) v = 0$$

then $\{v_1, v_2, \ldots, v_n, v\}$ is linearly dependent

Therefore $S \cup \{v\}$ is linearly dependent. (why?).

Conversely, let $S \cup \{v\}$ is linearly dependent

then there exists finite vectors $u_1, u_2, \ldots, u_m$ in $S \cup \{v\}$ such that for some nonzero scalars,

$$a_1 u_1 + a_2 u_2 + \ldots + a_m u_m = 0$$

then one of the u_i's is equal to v, if not then S become linearly dependent but S is linearly independent. Let say $u_1 = v$ for that $a_1 \neq 0$ (?)

$$\therefore \quad a_1 v + a_2 u_2 + \ldots + a_m u_m = 0$$

$$\Rightarrow \quad v = (-a_2 a_1^{-1}) u_2 + \ldots + (-a_m a_1^{-1}) u_m$$

And as $-a_2 a_1^{-1}, \ldots, -a_m a_1^{-1}$ are all scalars.

$$v \in span(S).$$

Section 1.6

Def^n:- A subset β of a vector space V is called basis of V if following holds:

(i) β is linearly independent subset of V.

(ii) β generates V i.e. span $(\beta) = V$.

Examples:-

(1.) Let $V = \{0\}$, then ϕ is a basis of V.

As span$(\phi) = \{0\}$.

(2.) Let $\beta = \{(1,0,0), (0,1,0), (1,0,0)\}$ be the subset of $\mathbb{R}^3$, then β is a basis of $\mathbb{R}^3$.

As (i) β is linearly independent

(ii) span$(\beta) = \mathbb{R}^3$. (Check).

(3.) Let $\beta = \{(1,1,0), (1,0,1), (0,1,1)\}$ be the subsets of $\mathbb{R}^3$ then β is also a basis of $\mathbb{R}^3$

As β is linearly independent

and span $(\beta) = \mathbb{R}^3$ (Check)

Remarks:- A vector space can have more than one basis (as shown). The basis given in example (2) is called standard basis of $\mathbb{R}^3$.

(4.) In F^n or $\mathbb{R}^n$, let

$B = \{e_1, e_2, -, e_n\}$, where

$e_1 = (1, 0, \ldots 0), e_2 = (0, 1, \ldots, 0)$ & $e_n = (0, 0, \ldots, 0, 1)$

be the subset of F^n, then B is the basis of F^n called <u>standard basis</u>. (Check)

(5.) Let

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

be the subset of $M_{2\times3}(\mathbb{R})$, then B is a basis of $M_{2\times3}(\mathbb{R})$.

As it can be easily checked that B is linearly independent subset of $M_{2\times3}(\mathbb{R})$ and B generates $M_{2\times3}(\mathbb{R})$. (Check)

(6.) In $M_{m\times n}(\mathbb{R})$, let E^{ij} denotes the matrix whose only nonzero entry is 1 and is in the i^{th} row and j^{th} column. Then $B = \{E^{ij} : 1 \le i \le m, 1 \le j \le n\}$ is a basis of $M_{m\times n}(\mathbb{R})$.

Thm 1.8 :- Let V be a vector space and $B = \{u_1, u_2, \ldots, u_n\}$ be a subset of V. Then B is a basis of V if and only if for each $v \in V$ can be uniquely expressed as a linear combination of vectors of B, i.e. can be expressed in the form

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

for unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Proof:- Let B be a basis of V, then

$$\text{Span}(B) = V$$

$\rightarrow$ Each vector $v \in V$ can be expressed as a linear combination of vectors of B, we only need to show that this representation is unique.

To show that ~~~~ let us assume

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \quad -① $$

and

$$v = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n \quad -②$$

for scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$.

then subtracting eq. ① and ②, we get

$$0 = (\alpha_1 - \beta_1) u_1 + (\alpha_2 - \beta_2) u_2 + \cdots + (\alpha_n - \beta_n) u_n$$

And since B is a basis of V.

B is a linearly independent subset of V.

$$\Rightarrow \alpha_1 - \beta_1 = 0, \ \alpha_2 - \beta_2 = 0, \ \ldots, \ \alpha_n - \beta_n = 0$$

$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n$

$\therefore$ u is uniquely expressible as a linear combination of vectors of B.

<u>Conversely</u>

Let every element of V can be uniquely expressed as a linear combination of vectors of B.

<u>Claim:-</u> B is a basis of V.

To prove above claim, we only need to show that B is a linearly independent subset of V.

Let there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ ~~not all~~ such that

$$0 = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \quad -(3)$$

But also

$$0 = 0.u_1 + 0.u_2 + \ldots + 0.u_n \quad -(4)$$

from (3) and (4), we get

$$\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$$

As $0 \in V$ and it can be uniquely expressed as linear combination of vectors of B.

$\therefore$ B is linearly independent.

Also $\mathrm{span}(B) = V$

$\therefore$ B is a basis of V.

Example

① As $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$. so every element of $\mathbb{R}^3$ has a unique representation as a linear combination of vectors of B.

So let $(x_1, x_2, x_3) \in \mathbb{R}^3$

then

$$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

is only representation of $(x_1, x_2, x_3) \in \mathbb{R}^3$.

② Let $B = \{1, x, x^2, \dots, x^n\}$ be the subset of $P_n(\mathbb{R})$ = set of polynomials of degree upto n.

then ~~B is~~ let $f(x) \in P_n(\mathbb{R})$

then $f(x) = a_0 + a_1 x + \dots + a_m x^m \quad (m \le n)$

$\Rightarrow f(x) = a_0 \cdot 1 + a_1 \cdot x + \dots + a_m \cdot x^m + 0x^{m+1} + \dots + 0 \cdot x^n$

this is the only representation of $f(x)$ as a linear combination of vectors of B.

$\therefore$ B is a basis of $P_n(\mathbb{R})$.

Note:- It can also be shown by def^n that B is a basis of $P_n(\mathbb{R})$. (Do it).

Upto now we have seen that Vector spaces can have basis. So question is does every vector space have basis. To answer it (partially), we have next theorem.

Thm 1.9 :- If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has finite basis.

[In other words, if a vector space is generated by a finite set, then this vector space must have basis containing finite vectors.]

Proof:- Let V be a vector space generated by a finite set S.

If $S = \phi$, then $span(S) = \{0\} = V$

$\therefore$ ϕ generates V and is a basis of V.

Also, if $S = \{0\}$, then $span(S) = \{0\} = V$

$\therefore$ S generates V and is a basis of V.

So, let us assume that S contains a non zero vector u_1.

then $\{u_1\}$ is a linearly independent set.

Continuing, if possible, choose $u_2, \ldots, u_k$ in S such that $\{u_1, u_2, \ldots, u_k\}$ is linearly independent subset of S and adding another vector in above set makes it linearly dependent.

(i.e. Choose $B = \{u_1, u_2, \ldots, u_n\}$ the largest possible linearly independent subset of S)

We now show that $B = \{u_1, u_2, \ldots, u_k\}$ is a basis of V.

Since B is linearly independent, we only need to show B generates V i.e. $\text{span}(B) = V$.

$$\begin{cases} \text{i.e. } \text{span}(B) = \text{span}(S) \\ \text{i.e. } \text{span}(S) \subseteq \text{span}(B) \quad \begin{pmatrix} \text{As } B \subseteq S \\ \text{span}(B) \subseteq \text{span}(S) \end{pmatrix} \\ \text{i.e. } S \subseteq \text{span}(B) \quad (\text{Using thm } 1.5) \end{cases}$$

$\therefore$ We need to show $S \subseteq \text{span}(B)$.

Let $v \in S$, if $v \in B$ then $v \in \text{span}(B)$.

and if $v \notin B$, then $B \cup \{v\}$ is linearly dependent [As B is largest linearly independent set).

Then By theorem 1.7,

$v \in \text{span}(B)$.

$\therefore \quad S \subseteq \text{span}(B)$

$\therefore \quad \text{span}(B) = V$

$\Rightarrow$ B is a basis of V. ∎

<u>Example</u>

Let $S = \{(2,-3,5), (8,-12,20), (1,0,-2), (0,2,-1), (7,2,0)\}$

then S generates $\mathbb{R}^3$ (Please do it).

We now get a subset of S which will be the basis of $\mathbb{R}^3$.

As in the proof of thm 1.9, we first choose a non zero vector, let we pick $(2,-3,5)$.

Now we have to choose other vectors from S such that these vectors along with $(2,-3,5)$ forms a linearly independent set.

Now let we pick $(8,-12,20)$, then

$$0 = 1\cdot(8,-12,20) - 4(2,-3,5)$$

$\Rightarrow$ $\{(2,-3,5), (8,-12,20)\}$ is not linearly independent.

Therefore, we can not pick $(8,-12,20)$.

Next we consider $(1,0,-2)$

and as $\{(1,0,-2), (2,-3,5)\}$ is linearly independent (check), so we have to pick $(1,0,-2)$ for basis.

Now next consider $(0,2,-1)$

We check whether $\{(2,-3,5), (1,0,-2), (0,2,-1)\}$ is linearly dependent or independent

To show let $\alpha_1, \alpha_2, \alpha_3$ be scalars such that

$$0 = \alpha_1(2,-3,5) + \alpha_2(1,0,-2) + \alpha_3(0,2,-1)$$

$$\Rightarrow \quad 0 = (2\alpha_1 + \alpha_2, -3\alpha_1 + \alpha_3, 5\alpha_1 - 2\alpha_2 - \alpha_3)$$

$$\Rightarrow \quad \begin{aligned} 2\alpha_1 + \alpha_2 &= 0 \\ -3\alpha_1 + \alpha_3 &= 0 \\ 5\alpha_1 - 2\alpha_2 - \alpha_3 &= 0 \end{aligned}$$

Solving above system of equations, we get

$\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$

$\therefore$ $\{(2,-3,5), (1,0,-2), (0,2,-1)\}$ is linearly independent set.

Next we can similarly check that $\{(2,-3,5), (1,0,-2), (0,2,-1), (7,2,0)\}$ is linearly dependent.

Therefore $\{(2,-3,5), (1,0,-2), (0,2,-1)\}$ is the largest linearly independent subset of S, and is a basis of $\mathbb{R}^3$. (from thm 1.7).

Qn:- Show that $B = \{(2,-3,5), (1,0,-2), (0,2,-1)\}$ is a basis of $\mathbb{R}^3$. [Do it by defn of basis].

Till now we have there may be more than one basis of a vector space. Consider $\mathbb{R}^3$ we have seen that $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$, also $\{(2,-3,5), (1,0,-2), (0,2,-1)\}$ is a basis of $\mathbb{R}^3$ and $\{(1,1,0), (1,0,1), (0,1,1)\}$ is also a basis of $\mathbb{R}^3$.

Looking closely ~~we~~ it is clear that every basis of $\mathbb{R}^3$ has exactly 3 ~~elem~~ vectors. So question is Do all basis of a vector space have same numbers of vectors. [The ans is Yes, for that we have following thm].

Thm 1.10 (Replacement theorem) :- Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \le n$ and there exist a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ generates V.

Proof:- We will prove this thm by mathematical induction on m.

If $m=0$, then $L=\phi$ so take $H=G$, then
$L \cup H = H$ generates V and $0 \le m$.
Which is the desired result.

Now suppose that the theorem is true for some integer $m \ge 0$ and we will prove the theorem for $m+1$.

Let $L=\{v_1, v_2, \ldots, v_{m+1}\}$ be a linearly independent subset of V containing $m+1$ vectors.

[We will show that $m+1 \le n$
and $\exists H \subseteq G$ s.t. H has $n-(m+1)$ vectors
and $L \cup H$ generates V]

then $L' = \{v_1, v_2, \ldots, v_m\}$ is linearly independent.

[$\because$ If $S_1 \subseteq S_2$ and S_2 is linearly independent then S_1 is also independent]

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Then by induction hypothesis, $m \leq n$ and there exists subset $\{u_1, u_2, \ldots, u_{n-m}\}$ of G containing $n-m$ vectors such that

$$\{v_1, v_2, \ldots, v_m\} \cup \{u_1, u_2, \ldots, u_{n-m}\} \text{ generates } V.$$

$\therefore$ for v_{m+1}, there exists scalar $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \ldots, \beta_{n-m}$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m + \beta_1 u_1 + \ldots + \beta_{n-m} u_{n-m} = v_{m+1} \quad - (1)$$

Note that $n-m > 0$, if not than $n-m=0$, then from eq. (1), we can write v_{m+1} as a linear combination of $\{v_1, v_2, \ldots, v_m\}$, but $L = \{v_1, v_2, \ldots, v_{m+1}\}$ is linearly independent.

$\therefore \quad n-m > 0$

$\Rightarrow \quad n > m$

$\Rightarrow \quad n \geq m+1 \quad - (2)$

~~Also some b_i, say b_1, is nonzero~~

Also some β_i, say β_1 is nonzero, if not we get the same contradiction.

As β_1 is nonzero, β_1^{-1} exists, so eq. (1) can be written as

$$u_1 = (-\beta_1^{-1}\alpha_1)v_1 + \ldots + (-\beta_1^{-1}\alpha_m)v_m + (+\beta_1^{-1})v_{m+1} + (-\beta_1^{-1}\beta_2)u_2 + \ldots + (-\beta_1^{-1}\beta_{n-m})u_{n-m} \quad - (3)$$

Let $H = \{u_2, u_3, \dots, u_{n-m}\}$

then H has $n-m-1 = n-(m+1)$ vectors

and from eq. ③, we can say that

$u_1 \in span(L \cup H)$

and as

$\{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_{n-m}\} \subseteq L \cup H$

$\Rightarrow$ $\{v_1, v_2, \dots, v_m, u_1, \dots, u_{n-m}\} \subseteq span(L \cup H)$

$\Rightarrow$ $span\{v_1, v_2, \dots, v_m, u_1, \dots, u_{n-m}\} \subseteq span(L \cup H)$

And as By induction hypothesis,

$span\{v_1, v_2, \dots, v_m, u_1, \dots, u_{n-m}\} = V$

$\therefore$ $span(L \cup H) = V$ — ④

from eq. ② and ④, we have

$n \geq m+1$ and have $H = \{u_2, \dots, u_{n-m}\}$ containing $n-(m+1)$ vectors such that $span(L \cup H) = V$.

which is desired.

$\therefore$ The theorem is true for $m+1$.

$\therefore$ By mathematical induction the theorem holds for all $m \geq 0$. ∎

Corollary 1:- Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.

Proof:- Suppose B is a finite basis for V having exactly n vectors.

Let B' be any other basis of V.

Suppose B' contains ~~fo~~ more than n vectors, select a subset S of B' containing n+1 vectors.

Then S is linearly independent subset of V containing n+1 vectors and B generate V. containing n vectors.

Then By Replacement thm (thm 1.10)

$n+1 \le n$, which is a contradiction.

$\therefore$ B' has finite no. of vectors not more than n say m.

$\therefore$ $m \le n$.

Now if $m < n$, we get the same contradiction. (?).

$\therefore$ $m = n$

$\Rightarrow$ B and B' have same no. of vectors.

$\underline{\text{Def}^n}$: A vector space is called <u>finite-dimensional</u> if it has a basis consisting of a finite no of vectors. The unique number of vectors in each basis for V is called the <u>dimension</u> of V and is denoted by dim(V).

A vector space that is not finite-dimensional is called <u>infinite-dimensional</u>.

<u>Examples</u>:-

① Let $V = \{0\}$, the basis of V is ϕ.

$\therefore$ $\dim(V) = 0$.

② Consider the vector space $\mathbb{R}^n$, then

$$B = \{(1,0,\ldots,0), (0,1,\ldots,0), \ldots, (0,\ldots,0,1)\}$$

is the basis of $\mathbb{R}^n$ and have n vectors.

$\therefore$ $\dim(\mathbb{R}^n) = n$.

③ Consider the vector space $P_n(\mathbb{R})$

then $B = \{1, x, x^2, \ldots, x^n\}$ is a basis of $P_n(\mathbb{R})$.

$\therefore$ $\dim(P_n(\mathbb{R})) = n+1$.

④ Consider the vector space $M_{m\times n}(F)$

then $B = \{E^{ij} : 1 \le i \le m, 1 \le j \le n\}$ where, E^{ij} is a $m \times n$ matrix whose only nonzero element is 1 and is at i^{th} row and j^{th} column., is a basis of $M_{m\times n}(F)$.

$\therefore \dim(M_{m\times n}(F)) = mn$

(5) Consider the set of complex nos. $\mathbb{C}$, then $\mathbb{C}$ is a vector space over $\mathbb{R}$ and $\beta = \{1, i\}$ is a basis of $\mathbb{C}$

$\therefore \dim(\mathbb{C}) = 2$.

(6) Consider $\mathbb{C}$ over the field $\mathbb{C}$, then $\{1\}$ is a basis of $\mathbb{C}$.

$\therefore \dim(\mathbb{C}) = 1$ if $\mathbb{C}$ is V.S. over $\mathbb{C}$.

(7) $P(\mathbb{R}) \rightarrow$ set of all polynomials, then $P(\mathbb{R})$ is a vector space over $\mathbb{R}$. and $\beta = \{1, x, x^2, \ldots\}$ is a basis of $P(\mathbb{R})$.

$\therefore$ $P(\mathbb{R})$ is infinite-dimensional.

<u>Corollary 2</u> :- Let V be a vector space with dimension n.

(a) Any finite generating set for V contains at least n vectors, and a generating set for V contains exactly n vectors is a basis for V.

(b) Any linearly independent subset of V that contains exactly n vectors is a basis for V.

(c) Every linearly independent subset of V can be extended to a basis for V.

Proof:- As $\dim(V) = n$

let β be a basis of V containing n vectors.

(a) let G be a finite generating set for V.

Then By thm 1.9, some subset ~~of~~ H of G is a basis for V.

and By corollary 1, H has exactly n vectors.

$\therefore$ G contains atleast n vectors. -

Now if G has ~~exco~~ exactly n vectors,

then again by thm 1.9, $\exists$ $H \subseteq G$ such that H is a basis of V.

Also, H has n vectors.

$\therefore$ $H = G$.

$\therefore$ G is a basis of V.

(b) let L be a linearly independent subset of V containing exactly n vectors.

Claim:- L is a basis for V.

As β generates V and L is linearly independent subset of V.

$\therefore$ By replacement theorem,

$\exists$ $H \subseteq \beta$ containing $n - n = 0$ vectors such that $L \cup H$ generates V.

and since $H = \phi$

$L \cup H = L$ generates V

$\therefore$ L is a basis for V.

(c) Let L be a linearly independent subset of V containing m vectors.

Claim:- L can be extended to a basis for V.

As $\mathcal{B}$ generates V. Therefore By Replacement theorem, $m \le n$, and $\exists$ $H \subseteq \mathcal{B}$ such that H has $n-m$ vectors and $L \cup H$ generates V.

then $L \cup H$ has atmost n vectors

By (a) part, $L \cup H$ has exactly n vectors and $L \cup H$ is a basis for V ■

which is desired.

Note:- What we get from above thm is that if vector space V has dimension n, and we have a generating subset of V containing n vectors, then that subset is basis for V.

Also if we have any linearly independent subset of V containing n vectors, than also that subset is basis for V.

Example

(1) As $\{x^2+3x-2, 2x^2+5x-3, -x^2-4x+4\}$ generates $P_2(R)$ — (Check)

$\therefore$ By ~~theorem~~ Corollary 2, this set is a basis for $P_2(R)$

(2) As $\{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$ is a linearly independent subset of R^4. (Check)

$\therefore$ By Corollary 2 (b), this set is a basis for R^4.

Question:- Do question 2 and 3, given in book at page no. 54.